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Comments on "Power-Law Falloff in the Kosterlitz–Thouless Phase of a Two-Dimensional Lattice Coulomb Gas"

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A simple argument is presented by which one can show that the critical inverse temperature β_c of a two-dimensional Coulomb gas (standard or hard-core) with activity z satisfies $\beta_c \leqslant \bar{\beta}$, where $\bar{\beta} = \bar{\beta}(z) \to (1+\sqrt{3})\, 8\pi/(3-\sqrt{3})$ in the low-activity limit. Previous results yield $\bar{\beta}(z) \to 24\pi$.

KEY WORDS: Coulomb gas; Kosterlitz–Thouless phase; power-law falloff; critical temperature; multiscale analysis.

In a recent paper, the present author, Marchetti *et al.*,[1] using multiscale analysis, proved that for a large class of two-dimensional lattice Coulomb gases and for all $\beta > \bar{\beta}$, depending on the system, the correlation between two external charges $\pm\xi$, such that $\bar{\eta} = \mathrm{dist}(\xi, Z - \{0\}) > 0$, has power-law falloff

$$G_\xi(x) \leqslant C/|x|^{\beta\theta\eta^2}$$

for some constants $C < \infty$ and $\theta > 0$, where $\eta = \mathrm{dist}(\xi, Z)$. Moreover, in the case of a hard-core or standard Coulomb gas with activity z, the authors proved that

$$\bar{\beta} \to \beta^*(\alpha) = 8\pi \frac{\alpha}{2-\alpha} \qquad \text{as} \quad z \searrow 0$$

where $2 > \alpha > 3/2$ is a parameter which controls the rapidity of changing scale, i.e., $L_{k+1} \cong L_k^\alpha$. So, the critical inverse temperature above which we

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get power decay is at most 24π in the low-activity limit (three times the conjectured critical temperature given by $\alpha = 1$ in the above equation).

Here I describe how to improve the lower bond for the scale parameter α by some small modifications of ref. 1. More precisely, I prove that the above-mentioned results are valid provided $2 > \alpha > (1 + \sqrt{3})/2$, which implies

$$\min \beta^* = \frac{(1 + \sqrt{3})\, 8\pi}{3 - \sqrt{3}} \approx 17.2\pi$$

The restriction on α comes from the "Energy estimate" of *isolated* charged distributions[2] (see Lemma 3.3 and Appendix B in ref. 1). In ref. 1 we end up with the following induction flow: if a density distribution ρ, localized on a box B_k of side L_k, is isolated inside a next scale box B_{k+1} of side $L_{k+1} = L_k^\alpha$, either [if its total charge $Q(\rho) \neq 0$] we apply a Spencer–MacBryan (SM)-type argument[3] in order to get extra falloff for the activity of ρ, or (if neutral) we put ρ separated. At certain point in the induction these isolated neutral charges will constitute a background of *neutral* and *sparse* weighed densities (α controlling the sparse condition). It turns out that to preserve the class of gases we are considering, the function $\hat{f}_j$ chosen for the (SM) argument must be constant over the densities belonging to the background. As a consequence, an extra factor, due the jump of the function around the support of these neutral densities, competes with wrong sign to the decay. Now, assuming $\hat{f}_j$ constant over all boxes where they are localized, the (SM) argument succeeds provided $\alpha > 3/2$. However, when the components of neutral densities are taken into account, we shall see that this bound can be sharpened.

Proposition. Let $\rho^{(k)}$ be a neutral weighed density localized on B_k composed by $(k-1, r + 2(\alpha - 1))$-admissible weighed densities $\{\rho_i^{(k-1)}, i = 1,..., n\}$. Suppose that in the next step $k \to k+1$, governed by Lemma 3.2 in ref. 1, $\rho^{(k)}$ continues isolated (i.e., $\rho^{(k+1)} = \rho^{(k)}$). If $n > n_0(\alpha) = 2 + \alpha$, then $\rho^{(k+1)}$ is $(k+1, r + (\alpha - 1))$-admissible.

Proof. Expression (3.6) in ref. 1 gives a bound L_{k-1}^{-nr} for the activity of $\rho^{(k)}$. One gets an entropic factor $L_k^{2(\alpha-1)}$ when $\rho^{(k)}$ becomes isolated in the next scale. Using the assumptions of Lemma 3.2 and the definition of (k, r)-admissible,[1] the activity of $\rho^{(k+1)}$ is bounded by

$$L_{k-1}^{-nr} L_k^{2(\alpha-1)} < L_{k-1}^{-(n-2)r} L_k^{-r} < L_{k+1}^{-r-2(\alpha-1)}$$

provided $r > 2\alpha^2(\alpha - 1)/(n - 2 + \alpha - \alpha^2)$. This condition is compatible with $r > 2\alpha(\alpha - 1)/(2 - \alpha)$, if $n > 2 + \alpha$.

It follows that isolated neutral weighed densities can be promoted to the next scale if the number of components is big enough. As a consequence, any distribution ρ localized on B_k, isolated on B_{k+1}, and belonging to the neutral background (sparse neutral ensemble in ref. 1) must have suport on n boxes of size L_{k-1} with $n = 1,..., [n_0(\alpha)]$. One is thus led to construct sparse neutral ensembles along the same lines of ref. 1, but now using a new admissible density.

Definition. A weighed charge density $\rho = (\rho, z(\rho))$ is (k, y, t)-*neutral admissible* if it is a neutral (k, y, t)-admissible charge density and a superposition of n $(k-1, y', t)$-admissible charge densities for some $y' \in B_k^{(k-1)}(y)$ with $n < n_0(\alpha)$.

I am now in position to discuss the main result. Since any (k, t)-neutral admissible density ρ is localized on a few boxes B_{k-1}, one gets for $(\hat{f}_j, \Delta\hat{f}_j)$ a perimeter factor of order L_{k-1} instead L_k due the jump of the function $\hat{f}_j$. This implies that the (SM) argument works if $L_{k-1} L_k^2/L_{k+1}^2 < \delta^k$, for some $\delta < 1$, which gives $\alpha > (1 + \sqrt{3})/2$.

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